Comment. Math.
Univ. St. Pauli
XXIV—1, 1975

On factorization theory of meromorphic functions

by

Fred GROSS

(Received April 10, 1975)

I. Introduction. The central problem of factorization theory can be stated intuitively as follows: Given a meromorphic function F, in how many ways can it be expressed as a composition of meromorphic functions $f_1, f_2, \cdots, f_n$. In other words, in how many ways can F be expressed in the form $F=f_1\circ f_2\circ\cdots\circ f_n$. Numerous special cases of this rather general problem have been studied by a variety of authors. Iteration theory for example has been extensively studied by numerous scholars including E. Schröder [94], P. Fatou [21], G. Julia [60], J. F. Ritt [90], H. Cremer [17], H. Töpfer [99], C. L. Siegel [96], J. Hadamard [48], U. T. Bödewadt [13], G. Szekeres [97], I. N. Baker [3, 6, 8], P. J. Myrberg [70], H. Kneser [62], H. Brolin [14], J. E. Wittington [105] and P. Bhattacharyya [12]. Properties of Commuting entire functions have also been investigated by a number of people among them: P. Fatou [20], G. Julia [59], V. Ganapathy Iyer [51], E. Jacobsthal [58] and I. N. Baker [4, 5]. The equation (1) $F=f_1\circ f_2$ has also been studied rather extensively. Some of the people who have been interested in this equation or certain special cases of it are G. Polya [99], P. Fatou [20, 22], H. Kneser [62], W. J. Thron [98], P. C. Rosenbloom [91, 92], I. N. Baker [1, 11], A. and C. Renyi [86, 87], F. Gross [29–47], M. Ozawa [78–84], R. Goldstein [25–27], E. Mues [69], C. C. Yang [44, 45, 101], W. H. J. Fuchs [24] and J. Clunie [15, 16].

The last problem on the solutions of equation (1) is actually much more than a special case of the general factorization problem. It seems, in fact, to include all of the more interesting aspects of the general problem. For this reason, we shall be concerned in the sequel only with this simpler version. Our primary objective is to give the reader a feel for the subject without burdening him with any of the lengthy proofs that are involved. We feel that this can be best accomplished by means of a few simple but revealing examples as well as a brief summary of the subject.

II. Definitions. In order to give more precise formulations of the basic problems of the theory, we introduce some necessary terminology.

Definition 1. A meromorphic function $h(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that f and g are meromorphic. The above representation of h is said to be a factorization of h.

Definition 2. A meromorphic function h is said to be prime if for every factorization of h at least one of the factors is bilinear.

Definition 3. A meromorphic function h is said to be pseudoprime if for every factorization of h at least one of the factors is rational.

Suppose that h has a factorization given by $h=f(g)$. Let L be any bilinear function. Then $h=f\circ L(L^{-1}\circ g)$ is another factorization of h. In order to get around this ambiguity, we introduce the following definition:

Definition 4. Two factorizations of h

$$h=f_1(g_1) \quad \text{and} \quad h=f_2(g_2)$$

are said to be equivalent if and only if there exists a bilinear function L such that $f_2=f_1\circ L$ and $g_2=L^{-1}\circ g_1$.

We now proceed to the next section, where we shall discuss a number of interesting examples.

III. Examples. In this section, we introduce a number of examples which serve as an excellent motivation for some of the problems we shall discuss.

1. Primes

(A) Polynomials—Every polynomial of prime degree is obviously prime. Most polynomials of composite degree are also prime. To see this, simply note that if $P=Q\circ T$, where P, Q, and T are polynomials of degrees n_P, n_Q and n_T, respectively, then $n_P=n_Q\cdot n_T$. Since the polynomials of degree n_P are essentially an $n_Q\cdot n_T$ parameter family while the composite functions are an n_Q+n_T parameter family, the assertion clearly follows.

(B) Rational Functions Which Are Not Polynomials—There are many examples of these. In particular, $P(z)/z$ is prime for every polynomial P with $P(0)\neq 0$.

(C) Transcendental Entire Primes—For any positive integer n, the function ze^{z^n} is prime. One sees this rather easily by considering the zeros and growth of all possible factors. A less trivial example is the function e^z+z. P. C. Rosenbloom [92] stated in 1952 that this function is prime. This fact was subsequently proved by the author [32]. More recently, Baker and the author proved that $e^z+P(z)$ is prime for any nonconstant polynomial $P(z)$. In fact, more is true. The author and

Yang [108] showed that Qe^z+g is prime for every nonzero polynomial Q and every nonconstant entire function g of order less than 1. Another interesting class of entire primes is the class $P(z)\sin\sqrt{z}/\sqrt{z}$, where $P(z)$ is any polynomial ($\not\equiv 0$). This class was suggested by Ozawa [83] and was recently proven to be prime by the author and Yang [45].

The last class of primes mentioned are special cases of a more gereral class of primes characterized in terms of their zeros. We now give an example of primes which are characterized by their Taylor coefficients. The author conjectured recently that for any sequence of prime integers $\{P_n\}_{n=1}^{\infty}$ any entire function of the form $\sum_{n=1}^{\infty} a_n z^{P_n}$ is prime. I. N. Baker* proved this when $\sum_{n=1}^{\infty} 1/(P_{n+1}-P_n)<\infty$.

(D) Transcendental Meromorphic Primes (which are not entire)— For any rational function $R(z)$ ($R(z)\not\equiv 0$), the functions $R(z)\sin\sqrt{z}/\sqrt{z}$ are prime (Gross and Yang [45]). Let $\wp$ be a Weierstrass pe-function with invariant $g_2\neq 0$, then its derivative $\wp'$ is prime (Mues [69]). Another interesting example is the Gamma function. Ozawa [83] was the first to conjecture that the Gamma function is prime. The proof was provided by the author and Yang [45].

2. Pseudoprimes Which Are Not Prime

Since all rational functions are pseudoprime, the only examples that come into question are transcendental. One interesting class of pseudoprimes is the class of periodic entire functions of exponential type (Gross [32]), i.e. the class of entire functions of the form $\sum_{-N}^{N} a_n e^{\lambda n z}$ where λ is a fixed constant. Another class worth mentioning is the class of functions $\wp'$ with $g_2=0$ (Gross [33]). The proofs that the above examples are pseudoprime depend on the fact that these functions are periodic as well as on their growth. Goldstein [25] provides examples which satisfy different conditions. He proved that any entire function f of finite order which satisfies for some finite a the deficiency condition $\delta(a,f)=1$ is pseudoprime. This includes, for example, functions like $e^{P(z)}$, where $P(z)$ is any polynomial.

3. Transcendental Entire Functions With Unique Factorization Into Primes

All primes obviously can be uniquely factored into nonlinear primes. We now give a nontrivial example: The function $F(z)=z^2e^{2z}$ has only one factorization into primes: $F(z)=f\circ g(z)$, where $f(w)=w^2$ and $g(z)=ze^z$ are primes.

4. Transcendental Entire Functions With Nonunique Factorization

The function $F(z)=\sum a_n z^{6n}$ has a factorization of the form $f_1(z^2)$ as well as one of the form $f_2(z^3)$ and these two factorizations are not equivalent. We may, however, be willing to broaden our definition of

* Not yet published.

equivalent by permitting the order of the factors to be ignored. Nevertheless, even in this broader sense there exist transcendental entire functions without unique factorization. For example, the function $F(z) = z^2e^{2z^2}$ has the two nonequivalent (even in the broader sense) factorizations $F(z)=f_i(g_i(z))$, $i=1, 2$, with $f_1(w)=w^2$, $g_1(z)=ze^{z^2}$ and $f_2(w)=we^{2w}$, $g_2(z)=z^2$.

5. Transcendental Entire Functions With Infinitely Many Nonequivalent Factorizations

For every nonzero integer n, one can write $e^z=(e^{z/n})^n$. In particular, w^p is a prime left factor of e^z for every prime integer p. $\cos z$ and $\sin z$ also have infinitely many possible factorizations as one can easily verify.

6. Transcendental Entire Functions Having a Factorization With Infinitely Many Factors

Kneser [62] exhibited an entire function f with the property that $f(cz)=e^f$ for a number c such that $e^c=c$. Thus $f(z)=\exp_n(f(z/c^n))$ and f has infinitely many left factors of the form $\exp_n(w)$, $n=1, 2, \cdots$, where $\exp_n(w)$ denotes the n-th iterate of the exponential function.

In the next section we shall look at some of the problems suggested by these examples.

IV. General Problems on Factorization. The examples in the previous section suggest a number of interesting problems. We list a few of these:

1. Characterize each of the following classes of meromorphic functions: (A) primes, (B) pseudoprimes, (C) functions that can be uniquely factored into nonlinear primes, and (D) functions that can be factored into finitely many primes (not necessarily uniquely).

2. Do there exist meromorphic functions with no prime factors? One might be tempted to use $f(t)=\exp_n(f(z/c^n))$ mentioned above, but there may exist other factorizations with prime factors.

3. Do there exist meromorphic functions with a nondenumerable number of factors?

4. Do there exist transcendental meromorphic functions with two distinct factorizations into primes each having a different numcer of prime factors? Ritt stated [89] that such functions do exist for rationals; he stated further that he would publish a proof of his assertion in a subsequent paper. The author has never found such a proof nor has he been able to prove or disprove Ritt's assertion. It seems therefore, at least to the author, that this problem remains open for rational functions as well.

None of the problems we have listed have been completely solved. Nevertheless, some work has been done on 1(A), (B) and (C), and we shall concentrate on these.

In our discussion it is convenient to discuss separate cases. The case of polynomials has been carefully studied by Ritt [89]. Ritt proved the following three theorems:

THEOREM 1. *Any two factorizations of a given polynomial into prime polynomials contain the same number of polynomials; the degrees of the polynomials in one factorization are the same as those in the other except perhaps for the order in which they occur.*

Before stating Theorem 2 we define equivalent factorization of polynomials explicitly.

Definition 5. Two factorizations of a polynomial $P(z)$ given by

$$P(z)=\phi_1(\phi_2\cdots(\phi_r(z))\cdots)$$

and

$$P(z)=\theta_1(\theta_2\cdots(\theta_r(z))\cdots) \quad (\phi_i, \theta_i \text{ Polynomials, } i=1, 2, \cdots, r)$$

are called equivalent if there exist $r-1$ polynomials of the first degree

$$\lambda_1(z), \lambda_2(z), \cdots, \lambda_{r-1}(z)$$

such that

$$\theta_1=\phi_1(\lambda_1), \ \theta_2=\lambda_1^{-1}\phi_2(\lambda_2), \cdots, \ \theta_r=\lambda_{r-1}^{-1}(\theta_r) .$$

Remark. If, in Definition 5, one replaces polynomials by meromorphic functions and linear polynomials by bilinear functions one has a generalization of Definition 4.

THEOREM 2. *Suppose that in a factorization of a polynomial $P(z)$ into prime polynomials*

$$P(z)=\phi_1(\phi_2\cdots(\phi_r(z)\cdots)) \tag{1}$$

we have for an adjacent pair of polynomials

$$\phi_i=\lambda_1\pi_1\lambda_2$$

and

$$\phi_{i+1}=\lambda_2^{-1}\pi_2\lambda_2 ,$$

where $\lambda_1(z)$, $\lambda_2(z)$ and $\lambda_3(z)$ are linear and where $\pi_1(z)$ and $\pi_2(z)$, of unequal degrees m and n, respectively, are of any of the following types:

(a) $\pi_1(z)=f_m(z)$ *and* $\pi_2(z)=f_n(z)$,
and where f_j is defined by

$$\cos ju=f_j(\cos u)$$

i.e. $f_n=2^{n-1}T_n$, *where T_n is the Chebyshev polynomial of degree n.*

(b) $\pi_1(z)=z^m$ *and* $\pi_2(z)=z^r g(z^m)$,

where $g(z)$ is any polynomial in z and r is an integer.
or
(c) $\pi_1(z)=(g(z))^n$ *and* $\pi_2(z)=z^n$.
Then there exists a factorization of $P(z)$ given by

$$P(z)=\phi_1(\phi_2\cdots(\phi_{i-1}(\theta_1(\theta_2(\phi_{i+2}\cdots(\phi_r)\cdots))))$$

in which $\theta_1(z)$ is of degree n and $\theta_2(z)$ is of degree m and this factorization is not equivalent to the factorization (1).

THEOREM 3. *If $P(z)$ has two nonequivalent factorizations, one can pass from either to a factorization equivalent to the other by repeated steps of the three types indicated in the previous theorem.*

No analogous theory exists for rational functions. As I mentioned earlier Ritt claimed that Theorem 1 has no analogue for rational functions. One can show, however, that every rational function can be factored into a finite number of prime factors and that one can generally readily determine whether a rational function is prime or composite (i.e. nonprime). Neither of the latter two statements seem to hold for transcendental functions. It is with the factorization of these transcendental functions that we shall primarily be concerned.

V. Some of the Techniques Used in the Theory.

There are various techniques that one could use to study factorizations of transcendental functions. Among the most useful are the following:

(1) The relationship between the zeros of the original function and those of its factors can be studied.

(2) One can compare the growth of the original function and that of its factors.

(3) One can apply the theory of conformal mapping.

(4) One can study the fixpoints of functions and their iterates. (z_0 is said to be a fixpoint of f if $f(z_0)=z_0$.)

(5) One can compare the Taylor coefficients of the original function and those of its factors.

We illustrate how some of these methods are used by looking at a simple example. Let Q be a given polynomial and let h be a given entire function. What can be said about the possible solutions g of

$$Q(g)=h? \tag{2}$$

By comparing growths one immediately concludes that g and h must be of the same order. By comparing the zeros of both sides of (2) one gets considerable information about where g attains the zeros of Q. Comparing the Taylor coefficients of g and h readily leads to the conclusion that there exist at most a finite number of solutions g. One can also easily show that if h has no fixpoints, no solutions exist unless Q is linear.

Now suppose we replace the polynomial Q in (2) by a transcendental meromorphic function f. i.e. consider the equation

$$f(g)=h\ , \tag{3}$$

where f and h are given meromorphic functions. The above discussion still applies. For example, Clunie showed [16] that comparing the growths of the two sides of (3) leads to $\lim_{r\to\infty} T(r, h)/T(r, g)=\infty$, $\overline{\lim}_{r\to\infty} T(r, h)/T(r, f)=\infty$, while for certain f and g $\underline{\lim}_{r\to\infty} T(r, h)/T(r, f)=0$. However, when f or g is of finite order or if f is entire, then $T(r, h)/T(r, f)=\infty$. Comparing Taylor coefficients in (3) yields the conclusion that there exist at most a denumerable number of solutions g. The other methods mentioned have also been used frequently in the literature to analyze (3), (e.g. see [11, 25, 31–33, 35, 41, 49, 69, 83]).

VI. Factorization of Periodic Functions. Let us return for a moment to equation (2). We noted that that equation had at most a finite number of solutions g. This observation leads us into a special area of interest; the factorization of periodic functions. We observe, in particular, that when h is periodic with period τ, then $Q(g(z+n\tau))=h(z)$ for $n=0, \pm 1, \pm 2, \cdots$. Thus, g would also have to be periodic. Hence, we see that if a periodic function has a left polynomial factor, then the corresponding right factor must be periodic. This fact was proved by A. and C. Renyi [86] and independently by the author [40]. Baker [11] and the author [32] independently generalized this result as follows: Let f be any entire function of lower order less than $\frac{1}{2}$. Then $f(g)$ is periodic if and only if g is. For f of order $\frac{1}{2}$, $\cos z=f(z^2)$ is a counterexample.

Baker [see 29] and Renyi [86] also proved independently that if P is a polynomial and f is entire then $f(P)$ can be periodic only if p is quadratic. This last result has been generalized recently in two directions. In one direction Fuchs and the author [43] proved the following result: Let $f(z)$ be a nonconstant meromorphic function and let $p(z)$ be a polynomial of degree n. The function $F(z)=f(p(z))$ cannot be periodic unless n has one of the values 1, 2, 3, 4, or 6. If $n=1$, then $F(z)$ can be any periodic meromorphic function. If $n=2$, then $F(z)$ is obtained by simple changes of variable from an even periodic function. If $n=3$, then F is an elliptic function and $F(z)=g((z+\alpha)^n)$ for a suitable meromorphic g and a complex α.

The second generalization was conjectured by C. Renyi in a letter to the author in 1964. It was recently proved by G. Halász [49]. Halász proved: If $f(z)$ is a transcendental entire function of order less than one, then $F(f(z))$ cannot be periodic provided $F(w)$, also entire, is not constant.

We conclude our discussion on factorization of periodic functions with some open problems. We stated earlier in our paper that the derivative of a Weierstrass elliptic function with invariant $g_2 \neq 0$ is prime. Do there exist entire periodic functions which are prime? Earlier in this section we noted that $P(g)$ is periodic if and only if g is, where P is a non-constant polynomial. One can readily verify that the assertion remains valid when P is replaced by a nonconstant rational function. What can be said about g when f is meromorphic and $f(g)$ is periodic? e.g., Does there exist a δ such that if $\rho(f)<\delta(\rho(f)$ denotes the order of $f)$ and $f(g)$ is periodic, then g is periodic? Can one find the largest such δ? If f is meromorphic, what conditions on g (g transcendental) assure that $f(g)$ is not periodic? Does there exist a δ such that $\rho(g)<\delta$ assures that $f(g)$ is not periodic? Can one find the largest such δ? A related question which the author posed in 1966 [30] is worth repeating here. Does there exist a nonperiodic entire function f such that $f(f)$ is periodic? One can list a number of other problems on factorization of periodic functions. The probles stated above, however, should give the reader a fairly good idea of some of the difficulties one encounters in this area.

VII. Uniqueness of Factorization and Commutativity. In our discussion above we have mentioned in particular, the problem of when the equation $f(g(z+\tau))=f(g(z))$ implies $g(z+\tau)=g(z)$. More generally one could look at the equation

$$f(g_1)=f(g_2) \tag{4}$$

where f is meromorphic and g_1 and g_2 are entire. We have already seen earlier that for f and g_1 fixed there exist at most denumerably many solutions g_2. When g_1 and g_2 are both polynomials growth considerations imply that they are both of the same degree, (when f is entire, g_1 is a polynomial whenever g_2 is) and more specifically, Baker and the author proved the following result [31]:

If $f(z)$ is a nonconstant entire function and $p(z)$ and $q(z)$ are nonconstant polynomials such that $f(p)=f(q)$, then either (i) there exist a root of unity λ and a constant β such that $p(z)=\lambda(q(z))+\beta$ or (ii) there exist a polynomial $r(z)$ and constants c, K such that $p(z)=(r(z))^2+K$, $q(z)=(r(z)+c)^2+K$. In case (i), either $\lambda=1$ and $f(z)$ is periodic with period β, or λ is a primitive j^{th} root of unity, $j>1$, in which case $f(z)$ has the form

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\eta)^{nj}, \quad \eta=\beta/(1-\lambda).$$

If $c\neq 0$ in case (ii), then $f(K+z^2)$ is an even periodic function of period c. Clearly all the cases mentioned above do occur. It is not known whether an analogous result holds for meromorphic functions f.

By a method similar to the one used by Baker and the author [31], L. Flatto [23] proved the following extension of the above:

Let P and Q be two polynomials of the same degree. Let $f(P)=g(Q)$, where f and g are two nonconstant entire functions. Then either (i) $P(z)=\lambda Q(z)+a$, where λ and a are constants or (ii) $P(z)=r^2(z)+a$ and $Q(z)=br^2(z)+cr(z)+d$, where $r(z)$ is a polynomial and b, c and d are constants with $b\neq 0$.

It is worth noting that these results related to uniqueness of factorization can be used in areas which, at least on the surface, seem to have little to do with problems of factorization. For example, Flatto [23] used his result mentioned above to prove the following interesting fact:

Let $p(t)$, $q(t)$ be polynomials with real coefficients such that the degree of p+degree of $q>0$. Let Γ denote the curve $x=p(t)$, $y=q(t)$, $-\infty<t<\infty$. Γ is a level curve of an everywhere harmonic function if and only if Γ is a part of a straight line or a parabola.

We have observed that the problem of finding solutions to equation (4) has bearing on the question of uniqueness of factorization. Another basic question which certainly should be mentioned in any discussion on uniqueness is the question of commutativity, i.e., if f and g are entire what can be said about the equation $f(g)=g(f)$? Some results on this problem are known. Given an entire function g, let $P(g)$ denote the family of all entire functions that commute with g. We now list some of the results known about $P(g)$:

A. (E. Jacobsthal [58]) Let $g(z)=b_0z^n+b_1z^{n-1}+\cdots+b_n$; $n\geqq 2$ $(b_0\neq 0)$. Then there exist at most n constants in $P(g)$, while for any positive integer m, there exist at most $n-1$ polynomials $f(z)$ of degree m in $P(g)$.

B. (Baker [5], Iyer [51]) If g is a nonconstant polynomial and $P(g)$ contains a transcendental $f(z)$, then $g(z)$ has the form $g(z)=ze^{2\pi mi/n}+b$, where b is a constant and m and n are integers. Conversely, to every pair of integers m, n and every complex constant b, there exists a transcendental $f(z)$ in $P(ze^{2\pi mi/n}+b)$.

An immediate consequence of this result is that for any polynomial g of degree $\geqq 2$, $P(g)$ is denumerable. Fatou originally conjectured that this was true for all nonlinear entire g. His conjecture was recently proved by Baker [7]. We state this result explicitly.

C. (Baker [7]) Let g be any nonlinear entire function. Then $P(g)$ is denumerable.

VIII. Fixpoints and Factorization. In this section we shall summarize some of the results on fixpoints which are closely related to our subject.

Fatou [21] originally stated that if f is entire and transcendental, then $f \circ f$ has infinitely many fixpoints. Subsequently, Rosenbloom [92] proved the following generalization of Fatou's assertion:

If f and g are entire transcendental functions, then one f or $f \circ g$ has infinitely many fixpoints.

The author [32] extended this result as follows:

If f is a transcendental meromorphic function and g and h are two transcendental entire functions, then one of the functions f, $f \circ g$, or $f \circ g \circ h$ has infinitely many fixpoints.

It is reasonable to conjecture more generally that if f is transcendental meromorphic and if g is any entire function other than a polynomial of degree $\leqq 2$, then $f \circ g$ has infinitely many fixpoints. When g is a polynomial the above assertion holds [44]. The conjecture is also true when f is entire and $\rho(f \circ g) < \infty$ [44]. When f is entire one can say a little more, for in this case the conjecture can be reformulated as follows: Let Q be any polynomial ($Q \not\equiv 0$) and let α be any entire function ($\alpha \not\equiv$constant), then $Qe^{\alpha} + z$ is prime. We have already seen above that $Qe^{P} + z$ is prime for any nonconstant polynomial P. Other special cases are also known. For example, when $H(z)$ is a periodic entire function of exponential type and nonconstant, then $e^{H(z)} + z$ is prime. Thus, in particular $\exp(\exp(z)) + z$ is prime. It is not known, however, whether $\exp(\exp(\exp(z))) + z$ is prime. More generally, it is not known whether $\exp_n(z) + z$ is prime for $n = 3, 4, 5, \cdots$.

IX. Concluding Remarks. Many important problems and results on commutativity fixpoint theory and other related areas not mentioned here but referred to in our introduction, play an important role in factorization theory. Baker's result C in Section VII, for example, is a consequence of his earlier work on fixpoints and iterates. It would, however, have been extremely difficult in such a short treatise to present the various connections and interrelationships between Nevanlinna theory, Wiman Valiron theory, the theory of conformal mapping, fixpoint theory, factorization theory and other aspects of functional equations without sacrificing the simplicity of our presentation. Many of these relationships, however, can be found in the literature referred to in our introduction. The author hopes that despite the brevity of this work, the reader has gained some insight into at least a few of the basic problems of factorization.

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Mathematics Research Center,
Naval Research Laboratory,
Washington, D. C. and
University of Maryland, Baltimore County.